

# A note on the vacant set of random walks on the hypercube and other regular graphs of high degree

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## Abstract

We consider a random walk on a  $d$ -regular graph  $G$  where  $d \rightarrow \infty$  and  $G$  satisfies certain conditions. Our prime example is the  $d$ -dimensional hypercube, which has  $n = 2^d$  vertices. We explore the likely component structure of the vacant set, i.e. the set of unvisited vertices. Let  $\Lambda(t)$  be the subgraph induced by the vacant set of the walk at step  $t$ . We show that if certain conditions are satisfied then the graph  $\Lambda(t)$  undergoes a phase transition at around  $t^* = n \log_e d$ . Our results are that if  $t \leq (1 - \varepsilon)t^*$  then w.h.p. as the number vertices  $n \rightarrow \infty$ , the size  $L_1(t)$  of the largest component satisfies  $L_1 \gg \log n$  whereas if  $t \geq (1 + \varepsilon)t^*$  then  $L_1(t) = o(\log n)$ .

## 1 Introduction

The problem we consider can be described as follows. We have a finite graph  $G = (V, E)$ , and a simple random walk  $\mathcal{W} = \mathcal{W}_u$  on  $G$ , starting at  $u \in V$ . In this walk, if  $\mathcal{W}(t)$  denotes the position of the walk after  $t$  steps, then  $\mathcal{W}(0) = u$  and if  $\mathcal{W}(t) = v$  then  $\mathcal{W}(t + 1)$  is equally likely to be any neighbour of  $v$ . We consider the likely component structure of the subgraph  $\Lambda(t)$  induced by the unvisited vertices of  $G$  at step  $t$  of the walk.

Initially all vertices  $V$  of  $G$  are unvisited or *vacant*. We regard unvisited vertices as colored *red*. When  $\mathcal{W}_u$  visits a vertex, the vertex is re-colored *blue*. Let  $\mathcal{W}_u(t)$  denote the position of  $\mathcal{W}_u$  at step  $t$ . Let  $\mathcal{B}_u(t) = \{\mathcal{W}_u(0), \mathcal{W}_u(1), \dots, \mathcal{W}_u(t)\}$  be the set of blue vertices at the end of step  $t$ , and  $\mathcal{R}_u(t) = V \setminus \mathcal{B}_u(t)$ . Let  $\Lambda_u(t) = G[\mathcal{R}_u(t)]$  be the subgraph of  $G$  induced by  $\mathcal{R}_u(t)$ . Initially  $\Lambda_u(0)$  is connected, unless  $u$  is a cut-vertex. As the walk continues,  $\Lambda_u(t)$  will shrink to the empty graph once every vertex has been visited. We wish to determine, as far as possible, the likely evolution of the component structure as  $t$  increases.

For several graph models, it has been shown that the component structure of  $\Lambda(t) = \Lambda_u(t)$  undergoes a phase transition of some sort. In this paper we add results for some new classes of graphs. What we expect to happen is that there is a time  $t^*$ , such that if  $t \geq (1 + \varepsilon)t^*$  then w.h.p. all components of  $\Lambda(t)$  are “small” and if  $t \leq (1 - \varepsilon)t^*$  then w.h.p.  $\Lambda(t)$  contains some “large” components. Here  $\varepsilon$  is some arbitrarily small positive constant and the meanings of small, large will be made clear.

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## 1.1 Previous work

We begin with the paper by Černý, Teixeira and Windisch [3]. They consider a sequence of  $n$ -vertex graphs  $G_n$  with the following properties:

- A1  $G_n$  is  $d$ -regular,  $3 \leq d = O(1)$ .
- A2 For any  $v \in V(G_n)$ , there is at most one cycle within distance  $\alpha \log_{d-1} n$  of  $v$  for some  $\alpha \in (0, 1)$ .
- A3 The second eigenvalue  $\lambda_2$  of the random walk transition matrix satisfies  $\lambda_2 \leq 1 - \beta$  for some constant  $\beta \in (0, 1)$ .

Let

$$t^* = \frac{d(d-1) \log(d-1)}{(d-2)^2} n. \quad (1)$$

In which case, it is shown in [3] that for  $t \leq (1 - \varepsilon)t^*$  there is w.h.p. a unique giant component in  $\Lambda(t)$  of size  $\Omega(n)$  and other components are all of size  $o(n)$ . Furthermore, if  $t \geq (1 + \varepsilon)t^*$  then all components of  $\Lambda(t)$  are of size  $O(\log n)$ .

The most natural class of graphs satisfying A1,A2,A3 are random  $d$ -regular graphs,  $3 \leq d = O(1)$ . For this class of graphs Cooper and Frieze [8] tightened the above results in the following ways. (i) they established the asymptotic size of the giant component for  $t \leq (1 - \varepsilon)t^*$ , and proved all other components have size  $O(\log n)$ ; (ii) they proved almost all small components are trees, and gave a detailed census of the number of trees of sizes  $O(\log n)$ . Subsequent to this work, Černý and Teixeira [4] built on the methodology of [8] and analysed the component structure at time  $t^*$  itself. More recently, for random  $d$ -regular graphs,  $3 \leq d = O(1)$ , Cooper and Frieze [9] determined the phase transition for a related structure, the *vacant net*, which by analogy with vacant set, they define as the subgraph induced by the unvisited edges of the graph  $G$ . Initially all edges are unvisited. The random walk *visits an edge* by making a transition using the edge.

In the paper [8], Cooper and Frieze also considered the class of Erdős-Reñyi random graphs  $G_{n,p}$  with edge probabilities  $p$  above the connectivity threshold  $p = \log n/n$ . For  $G_{n,p}$  where  $p = c \log n/n$ ,  $(c - 1) \log n \rightarrow \infty$ , they established that  $\Lambda(t)$  undergoes a phase transition around  $t^* = n \log \log n$ . For these graphs, at  $t_{-\varepsilon} = (1 - \varepsilon)t^*$  the size  $L_1$  of the largest component cannot be  $\Omega(n)$  since the vacant set has size  $|\mathcal{R}(t_\varepsilon)| = o(n)$  w.h.p. On the other hand it was shown that  $L_1 = \Omega(|\mathcal{R}(t_\varepsilon)|)$  w.h.p. More recently, Wassmer [15] found the phase transition in  $\Lambda(t)$  when the underlying graph is the giant component of  $G_{n,p}$ ,  $p = c/n$ ,  $c > 1$ .

There has also been a considerable amount of research on the  $d$ -dimensional grid  $\mathbb{Z}^d$  and the  $d$ -dimensional torus  $(\mathbb{Z}/n\mathbb{Z})^d$ . Here the results are less precise. Benjamini and Sznitman [2] and Windisch [16] investigated the structure of the vacant set of a random walk on a  $d$ -dimensional torus. The main focus of this work is to apply the method of random interacements. For toroidal grids of dimension  $d \geq 5$ , it is shown that there is a value  $t^+(d)$ , linear in  $n$ , above which the vacant set is sub-critical, and a value of  $t^-(d)$  below which the graph is super-critical. It is believed that there is a phase transition for  $d \geq 3$ . A recent monograph by Černý and Teixeira [5] summarizes the random interlacement methodology. The monograph also gives details for the vacant set of random  $r$ -regular graphs.

## 1.2 New results

In this note we consider certain types of  $d$ -regular graphs with  $n$  vertices, where  $d \rightarrow \infty$  with  $n$ . Our main example of interest is the hypercube  $Q_d$  which has  $n = 2^d$  vertices. The vertex set of the hypercube is sequences  $\{0, 1\}^d$  where two sequences are defined as adjacent iff they differ in exactly one coordinate. In order to be slightly more general, we identify those properties of the hypercube that underpin our results.

Given certain properties (listed below), we can show that w.h.p. the graph  $\Lambda(t)$  exhibits a change in component structure at around the time  $t^* = n \log d$  which is asymptotically equal to the expression in (1). We show that if  $t \leq t_{-\varepsilon} = (1 - \varepsilon)t^*$  then w.h.p. there are components in  $\Lambda(t)$  of size much larger than  $\log n$ , whereas if  $t \geq t_{\varepsilon} = (1 + \varepsilon)t^*$  then all components of  $\Lambda(t)$  are of size  $o(\log n)$ .

We use the notation  $\mathbf{Pr}(\mathcal{W}_x(t) = y)$  and  $P_x^t(y)$  for the probability that a random walk starting from vertex  $x$  is at vertex  $y$  at step  $t$ . If  $t$  is sufficiently large, so that the walk is very close to stationarity and the starting point  $x$  is arbitrary, we may also use the simplified notation  $\mathbf{Pr}(\mathcal{W}(t) = y)$ . Let  $\pi_v = d(v)/2m$  to denote the stationary probability of vertex  $v$ , where  $m = |E|$  is the number of edges of the graph  $G$  and  $d(v)$  is the degree of  $v$ . For regular graphs,  $\pi_v = 1/n$ . The rate of convergence of the walk is given by

$$|P_x^t(y) - \pi_y| \leq (\pi_y/\pi_x)^{1/2} \lambda^t, \quad (2)$$

where  $\lambda = \max(\lambda_2, |\lambda_n|)$  is the second largest eigenvalue of the transition matrix in absolute value. See for example, Lovasz [14] Theorem 5.1.

The hypercube  $Q_d$  is bipartite, and a simple random walk does not have a stationary distribution on bipartite graphs. To overcome this, we replace the simple random walk by a *lazy* walk, in which at each step there is a  $1/2$  probability of staying put. Let  $N_G(v)$  denote the neighbours of  $v$  in  $G$ , and  $d_G(v) = |N_G(v)|$ . The lazy walk  $\mathcal{W}$  has transition probabilities  $P_v^t(w)$  given by

$$P_v^t(w) = \begin{cases} \frac{1}{2} & w = v \\ \frac{1}{2d_G(v)} & w \in N_G(v) \\ 0 & \text{Otherwise} \end{cases}.$$

We can obtain the underlying simple random walk, which we refer to as the *speedy* walk, by ignoring the steps when the particle does not move. For large  $t$ , asymptotically half of the steps in the lazy walk will not result in a change of vertex. Therefore w.h.p. properties of the speedy walk after approximately  $t$  steps can be obtained from properties of the lazy walk after approximately  $2t$  steps.

The effect of making the walk lazy is to shift the eigenvalues of the simple random walk upwards so that, for the lazy walk  $\lambda = \lambda_2$ . We define a mixing time  $T$  for the lazy walk by

$$T = \min_{t \geq 1} \left\{ t : \left| P_x^t(y) - \frac{1}{n} \right| \leq \frac{1}{n^3} \right\}. \quad (3)$$

For the lazy walk, the spectral gap is  $1 - \lambda$ , so using this in (2), property **P1** implies that we can take  $T = O(d^{\rho_1} \log n)$  in (3).

### The graph properties we assume for our analysis

Let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ . For  $S \subset V$ , define  $N_G(S) = \{w \in V \setminus S : \exists v \in S \text{ s.t. } \{v, w\} \in E\}$ .

We assume that the graph  $G = (V, E)$  is  $d$ -regular, connected, and has the properties **P1**–**P4** listed below. The bounds in properties **P2**–**P4** are parameterised by the  $\varepsilon$  used to define  $t_{\pm\varepsilon}$  for the vacant set. We will point out later where we use these bounds, so that the reader can see their relevance.

**P1** The spectral gap for the lazy walk is  $\Omega(1/d^{\rho_1})$  for some constant  $0 < \rho_1 \leq 3$ . This implies that we can take  $T = O(d^{\rho_1} \log n)$  in (3), (see [13], Chapter 12).

**P2**  $(\log \log n)^{2/\varepsilon} \ll d = O\left(\frac{n}{\log n}\right)^{1/5}$ .

**P3** For  $u, v \in V$ , the graph distance  $\text{dist}_G(u, v)$  is the length of the shortest path from  $u$  to  $v$  in  $G$ . Let  $\nu(u, v)$  be the number of neighbours  $w$  of  $v$  for which  $\text{dist}_G(w, u) \leq \text{dist}_G(u, v)$ . Let  $\rho_2 = O(1)$ . Then for all  $u, v$  such that  $\text{dist}_G(u, v) \leq d^\varepsilon$ , we have  $\nu(u, v) \leq \rho_2 \text{dist}_G(u, v)$ .

**P4** For  $S \subseteq V$ , let  $e(S)$  denote the number of edges induced by  $S$ . If  $|S| = o(\log n)$ , then  $e(S) = o(d|S|)$ .

Properties **P1**–**P4** are various measures of expansion. Random regular graphs with degree  $d$  satisfying **P2** satisfy the other properties w.h.p. The hypercube  $Q_d$  satisfies these properties to a degree. Our results for the structure of the vacant set  $\Lambda(t)$  based on these properties are as follows.

**Theorem 1** *Let  $\varepsilon = \varepsilon(n)$  be a function such that  $\varepsilon \gg 1/\log d$ . Let  $t^* = n \log d$  and let  $t_{\pm\varepsilon} = (1 \pm \varepsilon)t^*$ . Let  $L_1(t)$  denote the size of the largest component in  $\Lambda(t)$ . At step  $t$  of the speedy walk, the following results for  $L_1(t)$  hold.*

(a) *If  $G$  satisfies **P1**, **P2**, **P3**, **P4**, and  $t \leq t_{-\varepsilon}$  then w.h.p.  $L_1(t) \geq e^{\Omega(d^{\varepsilon/2})}$ . Note that  $d^{\varepsilon/2}$  can be replaced by  $d^{\gamma\varepsilon}$  for any constant  $0 < \gamma < 1$ .*

(b) *If  $G$  satisfies **P1**, **P2**, **P3**, and  $t \geq t_{+\varepsilon}$  then w.h.p.  $L_1(t) = o(\log n)$ .*

We next prove that the hypercube  $Q_d$  satisfies Theorem 1(a),(b). To show this, we check that  $Q_d$  satisfies properties **P1**–**P4**. **P1** is satisfied with  $\rho_1 = 1$ , as the spectral gap for the lazy walk is  $\frac{2}{d}$  (see [13] page 162). **P2** is clearly satisfied, as  $d = \log_2 n$ . For **P3**, without loss of generality, let  $v = (0, 0, \dots, 0)$  and let  $u = (1, 1, \dots, 1, 0, \dots, 0)$  ( $k$  1's) be vertices of  $Q_d$ . There are exactly  $\nu(u, v) = k$  neighbours  $w$  of  $v$  which satisfy  $\text{dist}_G(u, w) \leq \text{dist}_G(u, v)$ , so we can take  $\rho_2 = 1$ . For **P4** we can use the edge isoperimetric inequality of Hart [11] which states that the number of edges between  $S$  and  $V - S$  is at least  $s(d - \log_2 s)$ , where  $|S| = s$ . This implies that  $S$  induces at most  $(s/2) \log_2 s$  edges. If  $s = o(d)$  then  $e(S) \leq (s/2) \log_2 s = o(ds)$ .

## 2 The main tools for our proofs

Given a graph  $G$  and random walk  $\mathcal{W}$ , let  $T$  be the mixing time given in (3). For a vertex  $v$ , let  $R_v = R_v(G)$  denote the expected number of visits to  $v$  by the speedy walk  $\mathcal{W}_v$  within  $T$  steps. Thus

$$R_v = \sum_{k=0}^T P_v^k(v). \quad (4)$$

Note that, as  $P_v^0(v) = 1$ ,  $R_v \geq 1$ .

Our main tool is a lemma (Lemma 1) that we have found very useful in analysing the cover time of various classes of random graphs. A general form of Lemma 1 as proved in [6] requires a certain technical condition to be satisfied. It was shown in [7] that provided  $R_v = O(1)$  for all  $v \in V$ , this condition is always true. We prove in Lemma 7 that if **P2** and **P3** hold, then  $R_v = 2 + O(1/d)$  for all  $v \in V$ . The probabilities given in Lemma 1 and Corollary 2 are with respect to the random walk.

**Lemma 1 (First visit lemma)** *Suppose that  $R_v = O(1)$  for  $v \in V$  and  $T\pi_v = o(1)$  and  $T\pi_v = \Omega(n^{-2})$ . Let*

$$f_t(u, v) = \Pr(t = \min \{\tau > T : \mathcal{W}_u(\tau) = v\})$$

*be the probability that the first visit to  $v$  after time  $T$  occurs at step  $t$ .*

*There exists*

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}, \quad (5)$$

*and constant  $K > 0$  such that for all  $t \geq T$ ,*

$$f_t(u, v) = (1 + O(T\pi_v)) \frac{p_v}{(1 + p_v)^{t+1}} + O(T\pi_v e^{-t/KT}). \quad (6)$$

□

**Corollary 2** *For  $t \geq T$  let  $\mathcal{A}_v(t)$  be the event that  $\mathcal{W}_u$  does not visit  $v$  at steps  $T, T+1, \dots, t$ . Then, under the assumptions of Lemma 1,*

$$\Pr(\mathcal{A}_v(t)) = \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} + O(T^2\pi_v e^{-t/KT}). \quad (7)$$

The result (7) follows by adding up (6) for  $s \geq t$ . □

**Remark 3** Provided  $p_v = o(1/T)$  and  $t \geq L$  where

$$L = 2KT \log n \quad (8)$$

then, as  $p_v = O(\pi_v)$ , bounds (6) and (7) can be written as

$$f_t(u, v) = (1 + O(T\pi_v)) p_v (1 - p_v)^t$$

and

$$\Pr(\mathcal{A}_v(t)) = (1 + O(T\pi_v)) (1 - p_v)^t$$

respectively. For the graphs we consider  $\pi_v = 1/n$ . From **P1**,  $T = O(d^{\rho_1} \log n)$  and from **P2**,  $d = O(n/\log n)^{1/4}$ . Thus for  $\rho_1 \leq 3$ ,  $p_v = o(1/T)$  as required.

## Contraction lemma

Let  $H = (V(H), E(H))$  be given. Let  $S$  be a subset of vertices of  $H$ . By contracting  $S$  to single vertex  $\gamma(S)$ , we form a multi-graph  $\Gamma = \Gamma(H, S) = (V', E')$  in which the set  $S$  is replaced by  $\gamma$ . The edges of  $H$ , including loops and multiple edges formed by contraction, are retained. Thus if  $(v, w) \in E(H)$  and  $v, w \notin S$  then  $(v, w) \in E'$ , whereas if  $v \in S$  and  $(v, w) \in E(H)$  then  $(\gamma, w) \in E'$ . This includes the case  $w \in S$  so that  $(\gamma, \gamma) \in E'$ . It follows that  $|E'| = |E(H)|$ . We prove in Lemma 4 that the probability of a first visit to  $S$  in  $H$  can be found (up to an additive error of  $O(|S|/n^3)$ ) from the probability of a first visit to  $\gamma$  in  $\Gamma$ .

Note that if  $T$  is a mixing time for  $\mathcal{W}$  in  $H$ , then  $T$  is a mixing time for the walk in  $\Gamma$ . We used the second eigenvalue  $\lambda_2(H) = \lambda$  of the lazy walk in (2) to obtain the mixing time bound in (3). It is proved in [1, Ch. 3], Corollary 27, that if a subset  $S$  of vertices is contracted to a single vertex, then the second eigenvalue of the transition matrix cannot increase. Thus  $\lambda_2(H) \geq \lambda_2(\Gamma)$ .

**Lemma 4** [6] *Let  $H = (V(H), E(H))$ , let  $S \subseteq V(H)$ , let  $\gamma(S)$  be the contraction of  $S$ , preserving edges, including loops and multiple edges. Let  $V' = V - S + \gamma$ , and let  $\Gamma(H) = (V', E')$ . Let  $\mathcal{W}_u$  be a random walk in  $H$  starting at  $u \notin S$ , and let  $\mathcal{X}_u$  be a random walk in  $\Gamma$ . Let  $T$  be a mixing time satisfying (3) in both  $H$  and  $\Gamma$ . For graphs  $G = H, \Gamma$ , let  $\mathcal{A}_w^G(t)$  be the event that in graph  $G$ , no visit was made to  $w$  at any step  $T \leq s \leq t$ . Then*

$$\Pr(\wedge_{v \in S} \mathcal{A}_v^H(t)) = \Pr(\mathcal{A}_\gamma^\Gamma(t)) + O(|S|/n^3).$$

### Proof

Note that  $|E(H)| = |E'| = m$ , say. Let  $W_x(j)$  (resp.  $X_x(j)$ ) be the position of walk  $\mathcal{W}_x$  (resp.  $\mathcal{X}_x(j)$ ) at step  $j$ . For graphs  $G = H, \Gamma$ , let  $P_u^s(x; G)$  be the  $s$  step transition probability for the corresponding walk to go from  $u$  to  $x$  in  $G$ .

$$\Pr(\mathcal{A}_\gamma^\Gamma(t)) = \sum_{x \neq \gamma} P_u^T(x; \Gamma) \Pr(X_x(s-T) \neq \gamma, T \leq s \leq t; \Gamma) \quad (9)$$

$$= \sum_{x \neq \gamma} \left( \frac{d(x)}{2m} + O(1/n^3) \right) \Pr(X_x(s-T) \neq \gamma, T \leq s \leq t; \Gamma) \quad (10)$$

$$= \sum_{x \notin S} (P_u^T(x; H) + O(1/n^3)) \Pr(W_x(s-T) \notin S, T \leq s \leq t; H) \quad (11)$$

$$\begin{aligned} &= \sum_{x \notin S} (\Pr(W_u(T) = x) \Pr(W_x(s-T) \notin S, T \leq s \leq t; H) + O(n^{-3})) \\ &= \Pr(W_u(t) \notin S, T \leq s \leq t; H) + O(|S|/n^3) \\ &= \Pr(\wedge_{v \in S} \mathcal{A}_v^H(t)) + O(|S|/n^3). \end{aligned} \quad (12)$$

In (9), if  $\mathcal{A}_\gamma^\Gamma(t)$  occurs then  $X_u(T) \neq \gamma$ . Given  $X_u(T) = x$ , by the Markov property  $X_u(s)$  is equivalent to the walk  $X_x(s-T)$ . After  $T$  steps, the walk  $X_u$  on  $\Gamma$  is close to stationarity. We use (3) to approximate  $P_u^T(x; \Gamma)$  by  $\pi_x = d(x)/2m = 1/n$  in (10). The second factor in equation (11) follows because there is a natural measure preserving map  $\phi$  between walks in  $H$  that start at  $x \notin S$  and avoid  $S$ , and walks in  $\Gamma$  that start at  $x \neq \gamma$  and avoid  $\gamma$ .  $\square$

**Remark 5** We use Lemma 4 throughout the rest of this paper, and often without further mention. Indeed most of the proofs rely on contracting some set  $S$  or other to a vertex  $\gamma(S)$ . In this case,

although a different graph  $\Gamma$ , and different walk  $\mathcal{X}$  are used to estimate the probability, provided

$$\frac{|S|}{n^3} = o(\Pr(\mathcal{A}_\gamma^\Gamma(t))),$$

the probability estimate we obtain for the walk  $\mathcal{W}$  in the base graph  $H$  is correct. If necessary, by increasing the mixing time  $T$  by a constant factor we can reduce the error term  $|S|/n^3$  to  $|S|/n^c$  for any  $c > 0$ .

## Visits to sets of vertices

Given the walk made a first visit to set of vertices  $S$ , we need the probability this first visit was to a given  $v \in S$ .

**Lemma 6** *Let  $S = \{v_1, \dots, v_k\}$  be a set of vertices of a graph  $G$ , such that Lemma 1 holds in  $G$  for all  $v \in S$ , and also for  $\gamma(S)$  in  $\Gamma(G)$ . For  $t \geq T$ , let  $\mathcal{E}_v = \mathcal{E}_v(t)$  be the event that the first visit to  $v$  after time  $T$  occurs at step  $t$ , (i.e.  $t = \min \{\tau > T : \mathcal{W}(\tau) = v\}$ ), and let  $\mathcal{E}_S = \cup_{v \in S} \mathcal{E}_v$ . Suppose  $t \geq 2(T + L)$  where  $L = 2KT \log n$ , where  $K > 0$  is some suitably large constant then for  $v \in S$*

$$\Pr(\mathcal{E}_v \mid \mathcal{E}_S) = \frac{p_v}{\sum_{w \in S} p_w} (1 + O(\xi)), \quad (13)$$

where  $\xi = L\pi_S$ , and  $p_w, w \in S$  are as defined in Lemma 1 for the walk on  $G$ .

**Proof** It is enough to prove the lemma for  $S = \{u, v\}$ , i.e. for two vertices, as vertex  $u$  can always be a contraction of a set. Specifically, if  $|S| > 2$  let  $u = \gamma(S \setminus \{v\})$ .

Write  $t$  as  $t = T + s + T + L$ , where  $s \geq L$ . Let  $\mathcal{A}_u$  be the event that  $\mathcal{W}(t) = u$ , but that  $\mathcal{W}(\sigma) \notin \{u, v\}$  for  $\sigma \in [T, s + T - 1]$ , and that  $\mathcal{W}(\sigma) \neq u$  for  $\sigma \in [s + 2T, t - 1]$ . Contract  $S$  to  $\gamma(S)$  and apply Corollary 2, Remark 3 and Lemma 4 to  $\gamma(S)$  in  $[T, T + s - 1]$ . The probability of no visit to  $S$  is  $(1 + O(T\pi_S))(1 - (p_u + p_v))^s$ . Next, apply Lemma 1 (and Remark 3) to  $u$  in  $[s + 2T, t] = [t - L, t]$ . The probability of a first visit to  $u$  is  $(1 + O(T\pi_u))(1 - p_u)^L p_u$ . Thus

$$\Pr(\mathcal{A}_u) \leq (1 + O(T\pi_S))(1 - (p_u + p_v))^s (1 - p_u)^L p_u. \quad (14)$$

Let  $\mathcal{B}_u$  be the event that  $\mathcal{W}(t) = u$  but  $\mathcal{W}(\sigma) \notin \{u, v\}$  for  $\sigma \in [T, t - 1]$ . Then  $\mathcal{B}_u \subseteq \mathcal{A}_u$  and so  $\Pr(\mathcal{B}_u) \leq \Pr(\mathcal{A}_u)$ . It follows from (14), and  $p_u L = O(\pi_S L)$  that

$$\Pr(\mathcal{B}_u) \leq p_u (1 - (p_u + p_v))^t (1 + O(\xi)). \quad (15)$$

However, by contracting  $S$  we have that the probability of a first visit to  $\gamma(S)$  at step  $t$  is

$$\Pr(\mathcal{B}_u \cup \mathcal{B}_v) = (1 + O(T\pi_S))(p_u + p_v)(1 - (p_u + p_v))^t. \quad (16)$$

From the above and (15)

$$\begin{aligned} \Pr(\mathcal{B}_v) &\geq \Pr(\mathcal{B}_u \cup \mathcal{B}_v) - \Pr(\mathcal{B}_u) \\ &\geq (1 - O(\xi))p_v(1 - (p_u + p_v))^t. \end{aligned} \quad (17)$$

Using (14), (16), (17) and  $\mathcal{E}_S = \mathcal{B}_u \cup \mathcal{B}_v$  the result follows from

$$\Pr(\mathcal{E}_v \mid \mathcal{E}_S) = \frac{\Pr(\mathcal{B}_v)}{\Pr(\mathcal{B}_u \cup \mathcal{B}_v)} \leq \frac{\Pr(\mathcal{A}_v)}{\Pr(\mathcal{B}_u \cup \mathcal{B}_v)}.$$

□

### 3 Proof of Theorem 1(a)

To apply the lemmas of the previous section we will need to estimate  $R_v$  as given by (4).

**Lemma 7** *If P2, P3 hold, then in the lazy walk, for any  $v \in V$*

(i)

$$R_v = 2 + \frac{2}{d} + O\left(\frac{1}{d^2}\right).$$

(ii) *Suppose  $\mathcal{W}(0)$  is at distance at least 2 from  $v$  (resp. at least 3 from  $v$ ). The probability  $\mathcal{W}$  visits  $N(v)$  within  $L = O(T \log n)$  steps is  $P(2, L) = O(1/d)$  (resp.  $P(3, L) = O(1/d^2)$ ).*

(iii) *Let  $C \subseteq N(v)$ . For a walk starting from  $u \in C$ , let  $R_C$  denote the expected number of returns to  $C$  during  $T$ . Then, in the lazy walk,  $R_C = 2 + O(1/d)$ .*

**Proof** *Proof of (i).* We write

$$R_v = 1 + \sum_{k=1}^T \frac{1}{2^k} + \sum_{k=0}^{T-1} \frac{1}{2^k} \sum_{w \in N_G(v)} \frac{1}{2d} R(w, T - k - 1),$$

where for  $w \in N_G(v)$ ,  $R(w, \tau)$  is the expected number of visits to  $v$  in  $\tau$  steps by  $\mathcal{W}_w$ .

For a lower bound,

$$R(w, \tau) \geq \sum_{j=0}^{\tau-1} \frac{1}{2^j} \frac{1}{2d} R_v = \frac{R_v}{d} \left(1 - \frac{1}{2^\tau}\right).$$

This is the probability that for some number of steps the walk loops at vertex  $w$ , and then moves to  $v$ , giving  $R_v$  expected returns to  $v$ . Thus

$$R_v \geq 2 - \frac{1}{2^{T+1}} + \frac{R_v}{2d} \sum_{k=0}^{T-1} \frac{1}{2^k} \left(1 - \frac{1}{2^{T-k-1}}\right),$$

so

$$R_v \geq 2 + \frac{2}{d} + O(1/d^2) - O(T/2^T).$$

As  $T \geq K \log n$  (see P1) we can assume that  $T2^{-T} = O(d^{-2})$ .

We next prove we can bound  $R(w, T)$  from above by

$$R(w, T) \leq R_v \left( \frac{1}{d} + O\left(\frac{1}{d^2}\right) \right). \quad (18)$$

Let  $N_G^i(v)$  is the set of vertices at distance  $i$  from  $v$  in  $G$ , and let  $R_i^* = \max_{w \in N_G^i(v)} R(w, T)$ . By definition  $R(w, T) \leq R_1^*$  and

$$R_1^* \leq \sum_{j \geq 0} \left( \frac{1}{2} + \frac{\rho_2}{2d} \right)^j \frac{1}{2d} R_v + \sum_{j \geq 0} \left( \frac{1}{2} + \frac{2\rho_2}{2d} \right)^j \frac{1}{2d} R_1^* + R_3^*. \quad (19)$$



The first summation term counts the case that for some number of steps the walk loops at a vertex of  $N_G^1(v)$ , or moves around in  $N_G^1(v)$ , which by **P3** has probability at most  $\rho_2/2d$ . At some point, the walk either moves to  $v$ , giving a  $R_v$  expected returns, or moves to  $N_G^2(v)$ . In the latter case, the second term counts moves back to  $N_G^1(v)$ , and the third term moves to  $N_G^3(v)$ , giving the  $R_3^*$  upper bound.

We next show that  $R_3^* = O(1/d^2)$ . Let  $\mathcal{X}$  be random walk on  $\{0, 1, \dots, \rho_3\}$ , with absorbing barriers at 0,  $\rho_3$ , and transition probabilities for  $\mathcal{X}(i)$  for  $0 < i < \rho_3$  given by

$$\mathcal{X}(i+1) = \begin{cases} \mathcal{X}(i) - 1 & \text{Probability } \frac{\rho_2 \rho_3}{d} \\ \mathcal{X}(i) & \text{Probability } \frac{1}{2} \\ \mathcal{X}(i) + 1 & \text{Probability } \frac{1}{2} - \frac{\rho_2 \rho_3}{d} \end{cases}.$$

Starting  $W_z$  from  $z \in N_G^3(v)$  and  $\mathcal{X} = \mathcal{X}_2$  from  $j = 3$ , we can couple  $W_z$  and  $\mathcal{X}$  so that  $\mathcal{X}$  is always as close to 0 as  $W_z$  is to  $v$ . Let  $u = W_z(t)$ . If  $\text{dist}(v, u) > \rho_3$  then  $\mathcal{X}$  is closer to  $v$ , where as if  $\text{dist}(v, u) \leq \rho_3$ , where  $\rho_3 \leq d^\varepsilon$ , then referring to **P3**,  $\nu(v, u) \leq \rho_2 \rho_3$ . The probability that  $W_z(t)$  moves towards  $v$  is at most the probability that  $\mathcal{X}$  moves towards 0.

For a random walk on  $0, 1, \dots, \ell$  starting from  $j = 0, 1, 2, \dots, \ell$  and with probabilities  $p, q$  of moving right or left respectively, it follows from XIV(2.4) of Feller [10] that the probability  $\pi_j$  of the walk visiting 0 before visiting  $\ell$  is

$$\pi_j = \frac{\xi^j - \xi^\ell}{1 - \xi^\ell} \leq 2\xi^j \quad (20)$$

where  $\xi = q/p$ . Thus for  $\mathcal{X}$  as given above,  $\xi = \rho_2 \ell / (d - 2\rho_2 \ell)$ .

To finish the proof of (i), we choose  $\ell = \rho_3 = \lceil d^\delta \rceil$ , for some  $\varepsilon/2 < \delta < \varepsilon$ . The probability  $\pi_3$  that  $\mathcal{X}$  reaches 0 before  $\rho_3$  is  $O(1/d^{3-3\delta}) = O(1/d^2)$ . Once the walk  $\mathcal{X}$  has reached  $\ell = \rho_3$ , we can restart it at  $\rho_3 - 1$ . The probability it reaches to the origin before a return to  $\rho_3$  is given by  $\pi_{\rho_3-1} = O(\xi^{\rho_3-1})$ . From **P1**,  $T = O(d^{\rho_1} \log n)$ , and we find

$$R_3^* \leq T\pi_{\rho_3-1} + \pi_3 = O(\log n d^{\rho_1+1-\rho_3(1-\delta)}) + O(1/d^2) = O(1/d^2).$$

For the last inequality, we used  $\delta > \varepsilon/2$  and **P2** to give

$$d^\delta \geq (\log \log n)^{2\delta/\varepsilon} > \log \log n.$$

*Proof of (ii).* Let  $C = \{v\} \cup N(v)$ . The property **P3** holds in  $G$  for any vertex at distance  $\ell \leq d^\varepsilon$  from  $v$ . Because moving closer to  $C$  implies moving closer to  $v$ , a vertex within distance  $\ell$  of  $v$  has at most  $\rho_2 \ell$  neighbours closer to  $C$ . If the walk starts at distance 2 from  $v$ , it either loops and/or moves within  $N_G^2(v)$ , or, conditional on making a transition away from  $N_G^2(v)$ , with probability  $O(2\rho_2/d)$  it moves to  $C$ , and with probability  $1 - O(1/d)$  moves to  $N_G^3(v)$ .

Assume the walk starts at a distance 3 from  $v$ . We define a graph  $\Gamma_C$  obtained from  $G$  by contracting the vertices in  $C$  to a single vertex  $\gamma_C$ . As explained before Lemma 4, we can still use the same mixing time  $T$ . If we replace  $v$  by  $\gamma_C$ , we can still use the coupling with the random walk  $\mathcal{X}$  on  $\{0, 1, \dots, \rho_3\}$ . As moving closer to  $\gamma_C$  means moving closer to  $v$ , choosing  $\rho_3 = \lfloor d^\varepsilon \rfloor - 1$ , it follows from **P3** as outlined above that the transition probabilities are correct. By the argument of part (i), the walk next moves to a distance  $\rho_3$  from  $\gamma_C$  with probability  $1 - O(1/d^2)$ . After this we use the argument of (i) as before. In conclusion, for a set  $C \subseteq N(v)$  and a walk which moves away from  $C$  to a distance 2 from  $v$ , (resp. distance 3 from  $v$ ) the probability of a return to  $\{v\} \cup N(v)$  within  $L$  steps is  $O(1/d)$  (resp.  $O(1/d^2)$ ).

*Proof of (iii).* Let  $C \subseteq \{v\} \cup N(v)$ . Contract  $C$  to  $\gamma_C$  as above. We claim that  $R_{\gamma_C} = 2 + O(\frac{1}{d})$ . The 2 comes from the laziness loop at each vertex and a factor of  $O(\rho_2/d)$  comes from possible loops at  $\gamma_C$  arising from  $G$ -edges inside  $C$ . If the walk moves to  $N_G^2(v)$ , then by (ii) the probability of a return to  $C$  is  $O(1/d)$ .  $\square$

### Analysis for $t \leq t_{-\varepsilon}$

Recall that  $t_{-\varepsilon} = (1 - \varepsilon)n \log d$ . Let  $U$  denote the set of vertices unvisited by the lazy walk in the time interval  $[1, 2t_{-\varepsilon}]$  and let  $U_0$  denote the set of vertices unvisited by the lazy walk in the time interval  $[T, 2t_{-\varepsilon}]$ . Note that  $|U_0 \setminus U| \leq T$ . Given Lemma 8 below holds, using **P1**, **P2** it follows that  $T = o(|U|)$  and thus  $|U| = (1 + o(1))|U_0|$ .

**Lemma 8** *w.h.p.*

$$|U_0| \sim \frac{n}{d^{1-\varepsilon}}.$$

**Proof** Fix a vertex  $v$ . Corollary 2 and Remark 3 tell us that

$$\Pr(v \in U_0) = \left(1 + O\left(\frac{T}{n}\right)\right) \exp\left\{-\frac{2t_{-\varepsilon}}{nR_v} + O\left(\frac{t_{-\varepsilon}}{n^2}\right)\right\} + O(e^{-\Omega(t_{-\varepsilon}/T)}). \quad (21)$$

By Lemma 7,  $R_v = 2 + \frac{2}{d} + O(\frac{1}{d^2})$ . This gives  $\Pr(v \in U_0) \sim d^{1-\varepsilon}$  and thus

$$\mathbf{E}|U_0| \sim \frac{n}{d^{1-\varepsilon}}.$$

Now consider a pair of vertices  $v, w$  at distance 5 or more in  $G$ . Let  $\Gamma_{vw}$  be obtained from  $G$  by contracting  $v, w$  to a single vertex  $\gamma_{vw}$ . Referring to Lemma 4 we have

$$\Pr(v, w \in U_0) = \Pr(\gamma_{vw} \in U_0) + O(1/n^3). \quad (22)$$

Working in  $\Gamma_{vw}$ , it follows more or less verbatim by using the arguments of Lemma 7(i) that  $R_{\gamma_{vw}} = 2 + \frac{2}{d} + O(\frac{1}{d^2})$ . As  $v, w$  are sufficiently far apart, only minor modifications are needed for the analysis of  $\mathcal{X}$ . Thus

$$\frac{2}{R_{\gamma_{vw}}} = \left(1 + O\left(\frac{1}{d^2}\right)\right) \left(\frac{1}{R_v} + \frac{1}{R_w}\right). \quad (23)$$

Using  $t_{-\varepsilon} = (1 - \varepsilon)n \log d$  in (21) it follows from (22) and (23) that

$$\Pr(v, w \in U_0) = \left(1 + O\left(\frac{\log d}{d^2}\right)\right) \Pr(v \in U_0) \Pr(w \in U_0) + O(1/n^3).$$

We prove concentration using the Chebychev inequality. Let  $X_{vw}$  be the indicator for  $v, w \in U_0$ . Let  $S$  be the set of pairs of vertices at distance at least 5, and let  $S'$  be the set of distinct pairs at distance at most 4. Then

$$\begin{aligned} \mathbf{E}|U_0|^2 &= \mathbf{E}|U_0| + \sum_{(v,w) \in S} \mathbf{E} X_{vw} + \sum_{(v,w) \in S'} \mathbf{E} X_{vw} \\ &\leq \mathbf{E}|U_0| + \left(1 + O\left(\frac{\log d}{d^2}\right)\right) \mathbf{E}|U_0|^2 + O(d^4 \mathbf{E}|U_0|). \end{aligned}$$

It follows from **P2** that  $d^4 = o(\mathbf{E}|U_0|)$ . Thus for some  $\omega$  tending to infinity

$$\Pr\left(|U_0| - \mathbf{E}|U_0| \leq \frac{\mathbf{E}|U_0|}{\sqrt{\omega}}\right) \leq O\left(\frac{\omega \log d}{d^2}\right) + O\left(\frac{\omega d^4}{\mathbf{E}|U_0|}\right) = o(1).$$

□

**Lemma 9** *A vertex is bad if it has fewer than  $d^\varepsilon/2$  neighbours in  $U$ . Let  $B$  denote the set of bad vertices. Then w.h.p.  $|B| \leq ne^{-d^\varepsilon/10}$ .*

**Proof** Fix a vertex  $v$  and denote  $N_G^1(v)$  by  $W = \{w_1, w_2, \dots, w_d\}$ . Let  $X = |W \cap U|$ . In the proof of Lemma 8 we showed that for a given vertex  $x$ ,  $\Pr(x \in U) = \tilde{p} \sim d^{-(1-\varepsilon)}$ . Thus  $\mathbf{E}X \sim d^\varepsilon$  and if  $X$  was distributed as  $\text{Bin}(d, \tilde{p})$  then it would be easy to show that

$$\Pr\left(X \leq \frac{1}{2}d^\varepsilon\right) \leq e^{-\Omega(d^\varepsilon)}. \quad (24)$$

The bound (24) is our target. We establish it is true, in spite of  $X$  not having a binomial distribution. For  $S \subseteq W$ , let  $\mathcal{A}_S = \{W \cap U = W \setminus S\}$ , i.e. exactly the vertices  $S$  of  $W$  are visited by the walk. So,

$$\Pr\left(X \leq \frac{1}{2}d^\varepsilon\right) = \sum_{D=d-d^\varepsilon/2}^d \sum_{\substack{S \subseteq W \\ |S|=D}} \Pr(\mathcal{A}_S). \quad (25)$$

If  $\mathcal{A}_S$  occurs then there is a sequence of times  $\mathbf{t} = (t_0 = 1 \leq t_1 < t_2 \leq \dots \leq t_D \leq t_{D+1} = 2t_{-\varepsilon})$  and a bijection  $f : S \rightarrow [D]$  such that for  $x \in S$  there is a first visit to  $w_x$  at time  $t_{f(x)}$ . Let  $\mathcal{B}(S, \mathbf{t})$  denote this event. For a sequence  $\mathbf{t}$ , let  $\Phi(\mathbf{t}) = \{i : |t_{i+1} - t_i| \leq L\}$ , where  $L = 2KT \log n$ . Let  $\mathcal{T}_h = \{\mathbf{t} : |\Phi(\mathbf{t})| = h\}$ . For  $h \geq 0$ , let

$$S_h = \sum_{\mathbf{t} \in \mathcal{T}_h} \Pr(\mathcal{B}(S, \mathbf{t})).$$

Then,

$$\Pr(\mathcal{A}_S) \leq \sum_{h=0}^D S_h. \quad (26)$$

The main content of the proof of this lemma will be to establish that

$$\Pr(\mathcal{A}_S) = O(1) (e^{-2pt-\varepsilon})^{(d-D)} (1 - e^{-2pt-\varepsilon})^D. \quad (27)$$

Given (25) and (27) we see that

$$\Pr\left(X \leq \frac{1}{2}d^\varepsilon\right) = O(1) \sum_{D \geq d-d^\varepsilon/2} \binom{d}{D} (e^{-2pt-\varepsilon})^{(d-D)} (1 - e^{-2pt-\varepsilon})^D.$$

The expected value of  $\text{Bin}(d, e^{-2pt-\varepsilon})$  is  $d^\varepsilon(1 + o(1))$ , so from the Hoeffding inequality,

$$\Pr\left(X \leq \frac{1}{2}d^\varepsilon\right) = O\left(e^{-d^\varepsilon/8}\right).$$

Thus, once we prove (27), the lemma follows from the Markov inequality.

**Proof of 27.** We begin with  $S_0$ . Our upper bound for  $S_0$  will contain some terms that should properly be assigned to some  $S_h, h > 0$ , but this is allowable as we proving an upper bound. We repeat this warning below. Let

$$p = \frac{1}{(2 + O(\frac{1}{d}))n}, \quad (28)$$

then we have

$$S_0 \leq D! \sum_{t_1 < t_2 \dots < t_D} \left( \prod_{i=1}^D \frac{(1 + O(T/n))p}{(1 + (d-i+1)p)^{t_i - t_{i-1}}} + o(e^{-\Omega(\frac{t_i - t_{i-1}}{T})}) \right) \times \left( \frac{1 + O(Td/n)}{(1 + (d-D)p)^{2t_{-e} - t_D}} + o(e^{-\Omega(\frac{t_{-e} - t_D}{T})}) \right). \quad (29)$$

**Proof of (29):** Assume for the moment that  $S = \{w_1, \dots, w_D\}$  and that  $f(w_i) = i$  for  $i = 1, 2, \dots, D$ . Let  $A_i = \{w_i, w_{i+1}, \dots, w_D\}$  for  $i = 1, 2, \dots, D$ . We assign times  $t_1, t_2, \dots, t_D$  to  $S$  in  $D!$  ways. Now consider a term

$$\Psi_i = \frac{(1 + O(T/n))p}{(1 + (d-i+1)p)^{t_i - t_{i-1}}} + o(e^{-\Omega((t_i - t_{i-1})/T)}). \quad (30)$$

We claim this is an estimate of the probability there are no visits to  $w_i, \dots, w_D$  during  $[t_{i-1} + T, t_i - 1]$  followed by a first visit to  $w_i$  at  $t_i$ . If so, it is also an upper bound for the probability there is no visit to  $w_i, \dots, w_D$  during  $[t_{i-1} + 1, t_i - 1]$  followed by a visit to  $w_i$  at  $t_i$ . This bound hold regardless of the first  $t_{i-1}$  steps of the walk. In fact this bound allows for visits to  $w_i, w_{i+1}, \dots, w_D$  during the time interval  $[t_{i-1} + 1, t_{i-1} + T - 1]$ , but this is allowable as  $\Psi_i$  is an upper bound. Thus some terms properly attributed to  $S_h, h > 0$  are overcounted.

To prove (30), define a graph  $\Gamma_{A_i}$  obtained from  $G$  by contracting the vertices in  $A_i$  to a single vertex  $\gamma_{A_i}$ . The mixing time  $T$  does not increase, as explained above Lemma 4. We also have  $R_{\gamma_{A_i}} \leq 2 + O(\frac{1}{d})$ . For this, we again follow the proof of Lemma 7. The 2 comes from the laziness loop at each vertex and the  $O(\frac{1}{d})$  comes from possible loops at  $\gamma_{A_i}$  arising from cases where there are  $G$ -edges inside  $A_i$ . We apply the same argument as in Lemma 7. We can use the random walk  $\mathcal{X}$  because a vertex  $z \neq \gamma_{A_i}$  and within distance  $\rho_3 - 1$  of  $\gamma_{A_i}$  has at most  $\rho_2 \rho_3$  neighbours closer to  $\gamma_{A_i}$ . This is because moving closer to  $\gamma_{A_i}$  implies moving closer to  $A_i \subseteq W$ , and hence to  $v$ . Apply **P3** to  $\{z, v\}$ .

By Lemma 1, the probability  $t_i$  is the time of a first visit to  $\gamma_{A_i}$  in  $[t_{i-1} + T, t_i]$  can be expressed as  $(d-i+1)\Psi_i$ . Given a first visit has been made to  $A_i$ , we need the probability that this first visit was made to a given  $v \in A_i$ . Lemma 6 gives the answer. The  $p_{w_j}, j = i, \dots, D$  used in Lemma 6 are given by (28). This establishes (30).

The final term in (29), given by  $\frac{1 + O(Td/n)}{(1 + (d-D)p)^{2t_{-e} - t_D}} + o(e^{-\Omega((t_{-e} - t_D)/T)})$  bounds the probability that the vertices in  $\{w_{D+1}, \dots, w_d\}$  are not visited in the interval  $[t_D, 2t_{-e}]$ . We use the first part of the argument for  $\Psi_i$  to validate this.

**End of proof of (29).**

The next step is to evaluate (29). Considering (30), the term  $\frac{p}{(1 + (d-i+1)p)^{t_i - t_{i-1}}} = \Omega((1/n)e^{(t_i - t_{i-1})/n})$ , whereas the term  $o(e^{-\Omega((t_i - t_{i-1})/T)}) = o(e^{(t_i - t_{i-1})/T})$ . As  $t_i - t_{i-1} \geq L = KT \log n$  the latter term

can be absorbed into the  $O(d^{-1})$  in the definition of  $p$ . Furthermore,

$$\frac{1}{1 + (d - i + 1)p} = \exp \left\{ -(d - i + 1)p + O\left(\frac{d^2}{n^2}\right) \right\}.$$

Noting that

$$\sum_{i=1}^{D+1} (d - i + 1)(t_i - t_{i-1}) = (d - D)t_{D+1} + (t_1 + \cdots + t_D),$$

we can write

$$\begin{aligned} S_0 &\leq 2D!p^D \sum_{t_1 < t_2 \cdots < t_D} \exp \left\{ -p \sum_{i=1}^{D+1} (d - i + 1)(t_i - t_{i-1}) \right\} \\ &= 2D!p^D e^{-2(d-D)pt-\varepsilon} \sum_{t_1 < t_2 \cdots < t_D} \exp \left\{ -p \sum_{i=1}^D t_i \right\} \\ &\leq 2e^{-2(d-D)pt-\varepsilon} \left( p \sum_{t=1}^{2t-\varepsilon} e^{-pt} \right)^D \\ &\leq 3e^{-2(d-D)pt-\varepsilon} \left( p \int_{t=0}^{2t-\varepsilon} e^{-pt} dt \right)^D \\ &= 3e^{-2(d-D)pt-\varepsilon} (1 - e^{-2pt-\varepsilon})^D \end{aligned} \tag{31}$$

We next show that  $S_1, S_2, \dots, S_D$  are not much larger in total than  $S_0$ .

We say a visit to vertex  $u$  is  $T$ -distinct, if it occurs at least  $T$  steps after a previous  $T$ -distinct visit, or from the start of the walk. Thus if  $\mathcal{W}(t) = u$ , and this visit is  $T$ -distinct, the next  $T$ -distinct visit to  $u$  will be at the first step  $s \geq t + T$  such that  $\mathcal{W}(s) = u$ . Once a  $T$ -distinct visit has taken place, several *secondary visits* to the vertex  $u$  may occur within the next  $T - 1$  steps, and thus before the next  $T$ -distinct visit. We will consider such secondary visits separately in our proof.

We consider the case  $t_i - t_{i-1} \leq L$  in two parts, namely  $t_i - t_{i-1} < T$ , and  $T \leq t_i - t_{i-1} \leq L$ . The first case is for *secondary visits*, and the second case *close (together) visits*. These require a separate analysis.

Given  $\mathbf{t} = (t_1, \dots, t_D)$  for arbitrary  $D \leq d$ , let  $Z \geq D - k$  be an upper bound on the total number of secondary visits to  $W = N(v)$  occurring as a result of  $k \leq D$  first visits to  $W$  which are  $T$ -distinct. Let  $N_2(v)$  denote the set of vertices at distance 2 from  $v$ . Then

$$Z(\mathbf{t}) = N_1 + \cdots + N_k$$

where  $N_i$  are the number of secondary visits to  $W = N(v)$  (i.e. returns to  $W$  via  $\{v\} \cup N_2(v)$ ) which occur during  $[t_i, t_i + T]$ ,  $i = 1, \dots, k$ .

The values of  $N_i$  are independent and geometrically distributed with failure probability  $O(1/d)$ . From  $W = N(v)$  the particle moves to  $\{v\} \cup N(v)$  with probability  $O(1/d)$ , (this follows from **P3**). Otherwise the particle moves to distance 2 away from  $v$  with probability  $1 - O(1/d)$ , and we can use the value of  $P(2, T) = O(1/d)$  from Lemma 7(ii). For any  $D \leq d$ , the probability  $\hat{P}(\ell)$  of at least  $\ell$  secondary visits is

$$\hat{P}(\ell) = \binom{D + \ell - 1}{\ell} \left( \frac{O(1)}{d} \right)^\ell \leq \left( \frac{O(1)D}{\ell d} \right)^\ell \leq \left( \frac{O(1)}{\ell} \right)^\ell = e^{-\Theta(\varepsilon d^\varepsilon \log d)},$$

on choosing  $\ell = d^\varepsilon/100$ . Provided  $\varepsilon \gg 1/\log d$ , the probability of at least  $d^\varepsilon/100$  secondary visits to  $W$  is  $o(e^{-d^\varepsilon})$ .

We next consider close together visits. For convenience, replace  $D$  by  $D' = D - Z$  i.e. remove any entries in  $\mathbf{t}$  corresponding to secondary visits. Let  $h$  count those  $T$ -distinct first visits which are close together i.e.  $T \leq t_i - t_{i-1} \leq L$ . After  $t \geq T$  steps, the distribution of the walk is close to stationary, so the probability that the walk is within distance 2 of vertex  $v$  is  $O(d^2/n)$ . If the walk is at least distance 3 from  $v$ , by Lemma 7(ii) the probability of a visit to  $W = N(v)$  in  $L$  steps is at most  $P(3, L) = O(1/d^2)$ . It follows that, independently of any previous ones, each close visit has probability  $O(d^2/n) + O(1/d^2) = O(1/d^2)$ , assuming  $d = o(n^{1/4})$  (see **P2**).

To bound  $S_h$  we note that the remaining  $k = D - h$  first visits are ‘well spaced’ i.e.  $L \leq t_i - t_{i-1}$ . There are  $\binom{D-1}{h}$  ways to assign the  $h$  ‘close together’ events to the  $k = D - h$  ‘well spaced’ ones. To do so, we choose an allocation  $n_1, n_2, \dots, n_k \geq 0$  such that  $n_1 + n_2 + \dots + n_k = h$ .

Note that  $S_0 = S_0(D)$  so changing  $D$  to  $D - h$ , for  $h \geq 1$ , from (31) we have

$$S_h(D) \leq S_0(D-h) \binom{D-1}{h} \left( \frac{O(1)}{d^2} \right)^h \leq S_0(D) \left( \frac{e^{2pt-\varepsilon}}{1 - e^{-2pt-\varepsilon}} \right)^h \left( \frac{O(D)}{hd^2} \right)^h \leq S_0(D) \left( \frac{O(d^{-\varepsilon})}{h} \right)^h. \quad (32)$$

The value of  $p$  is from (28), and  $t_{-\varepsilon} = (1 - \varepsilon)n \log d$ . Inequality (32), along with (31) completes the proof of (27), and the lemma follows.  $\square$

We can now easily show that w.h.p. at time  $2t_{-\varepsilon}$ , there is a component of size much larger than  $\log n$ .

**Lemma 10** *W.h.p. the graph induced by unvisited vertices contains a component of size at least  $e^{\Omega(d^{\varepsilon/2})}$ .*

**Proof** Let  $n_0 = \frac{n}{5(ed^{1-\varepsilon/2})^{d^{\varepsilon/2}} d^{1-\varepsilon}}$ . We begin by greedily choosing  $v_1, v_2, \dots, v_{n_0} \in U$  such that  $v_i, v_j$  are at distance greater than  $d^{\varepsilon/2}$ . This is easily done, because there are  $1 + \binom{d}{1} + \binom{d}{2} + \dots + \binom{d}{d^{\varepsilon/2}} < 2\binom{d}{d^{\varepsilon/2}} \leq 2(ed^{1-\varepsilon/2})^{d^{\varepsilon/2}}$  vertices within distance  $d^{\varepsilon/2}$  of any given vertex. Having chosen  $v_1, v_2, \dots, v_k$ ,  $k \leq n_0$ , there will w.h.p. be at least  $\frac{n}{2d^{1-\varepsilon}} - 2k(ed^{1-\varepsilon/2})^{d^{\varepsilon/2}} > 0$  choices for  $v_{k+1}$ . For each  $i$  let  $V_i$  denote the set of vertices within distance  $d^{\varepsilon/2}$  of  $v_i$ . The  $V_i$  are disjoint and so from Lemma 9 there are w.h.p. at least  $n_0 - ne^{-d^{\varepsilon}/10} > 0$  indices  $i$  such that  $V_i \cap B = \emptyset$ .

Choose  $i$  such that  $V_i \cap B = \emptyset$ . From  $v_i$  we can do breadth first search, but only including vertices in  $U$ . If  $L_r$  denotes the  $r$ th level of this search where  $L_0 = \{v_i\}$  then we see that  $|L_{r+1}| \geq \frac{d^\varepsilon |L_r|}{2\rho_2(r+1)}$ . Thus  $V_i$  contains a component of size at least

$$\sum_{i=0}^{d^{\varepsilon/2}/2} \binom{d^\varepsilon/2}{i} \frac{1}{(2\rho_2)^i} = e^{\Omega(d^{\varepsilon/2})}.$$

$\square$

## 4 Proof of Theorem 1(b)

Let

$$s = \frac{2 \log n}{\varepsilon \log d} = o(\log n).$$

We will show that w.h.p. there is no component of size  $s$  or more at time  $t \geq 2t_{+\varepsilon}$  in  $\Gamma(t)$ , with respect to the lazy walk.

**Lemma 11** *For  $v \in V$  there are at most  $(ed)^{s-1}$  sets  $S$  such that (i)  $v \in S$ , (ii)  $|S| = s$  and (iii)  $G[S]$  is connected.*

**Proof** The number of such sets is bounded by the number of distinct  $s$ -vertex trees which are rooted at  $v$ . This in turn is bounded by the number of distinct  $d$ -ary rooted trees with  $s$  vertices. This is equal to  $\binom{ds}{s}/((d-1)s+1)$ , see Knuth [12].  $\square$

We fix a set  $S$  of size  $s$  that induces a connected subgraph of  $G$ . To estimate the probability that  $S$  is unvisited at time  $t \geq 2t_{+\varepsilon}$  we contract  $S$  to a vertex  $\gamma_S$  as in the proofs of Lemmas 8 and 9. We need to estimate the probability that  $\gamma_S$  is unvisited by a lazy random walk on the associated graph  $\Gamma_S$  during the time interval  $[T, 2t_{+\varepsilon}]$ . For this we need to prove

**Lemma 12**  $R_{\gamma_S} = 2 + o(1)$ .

**Proof** Let  $e(S)$  denote the number of edges contained in  $S$ . It follows from **P4** that  $e(S) = o(ds)$ . This means that  $\gamma_S$  has degree  $ds$ , of which  $o(ds)$  comes from loops associated with internal edges of  $S$ . It then follows that when the walk on  $\Gamma_S$  is at  $\gamma_S$  then it leaves  $\gamma_S$  with probability  $\frac{1}{2} - o(1)$ . It is then straightforward to use the argument of Lemma 7 to finish the proof of the lemma.  $\square$

Using Lemma 11 and Lemma 12 we see that if  $p_\gamma = \frac{(1+o(1))s}{2}$  then

$$\begin{aligned} \Pr(\text{there exists a component of size } s) &\leq n(ed)^{s-1} \left( \frac{1 + O(Ts/n)}{(1 + p_\gamma)^{2t_{+\varepsilon}}} + O(T^2 s e^{-\Omega(t_{+\varepsilon}/T)}) \right) \\ &\leq 2n(ed \cdot e^{-(1-o(1))(1+\varepsilon)\log d})^s \\ &\leq 2nd^{-2\varepsilon s/3} = o(1). \end{aligned}$$

$\square$

## References

- [1] D. Aldous and J. Fill. Reversible Markov Chains and Random Walks on Graphs, <http://stat-www.berkeley.edu/pub/users/aldous/RWG/book.html>.
- [2] I. Benjamini and A. Sznitman, Giant component and vacant set for random walk on a discrete torus, *J. Eur. Math. Soc.*, 10 (2008) 1–40.
- [3] J. Černý, A. Teixeira and D. Windisch, Giant vacant component left by a random walk in a random  $d$ -regular graph. *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, 47 (2011) 929–968.
- [4] J. Černý and A. Teixeira, Critical window for the vacant set left by random walk on random regular graphs, *Random Structures and Algorithms*, 43 (2013) 313–337.

- [5] J. Černý and A. Teixeira, From random walk trajectories to random interlacements, *Sociedade Brasileira de Matemática, Ensaios Matemáticos*, 23 (2012) 1–78.
- [6] C. Cooper and A.M. Frieze, The cover time of the giant component of a random graph, *Random Structures and Algorithms*, 32 (2008) 401–439.
- [7] C. Cooper, A.M. Frieze and T. Radzik, The cover time of random walks on random uniform hypergraphs, *Theoretical Computer Science*, 509 (2013) 51–69.
- [8] C. Cooper and A.M. Frieze, Component structure induced by a random walk on a random graph, *Random structures and Algorithms*, 42 (2013) 135–158.
- [9] C. Cooper and A.M. Frieze, Vacant sets and vacant nets: Component structures induced by a random walk. [arXiv:1404.4403](#) (2014).
- [10] W. Feller, *An Introduction to Probability Theory, Volume I*, (Second edition) Wiley (1960).
- [11] S. Hart, A note on the edges of the  $n$ -cube, *Discrete Mathematics*, 14 (1976) 157–163.
- [12] D.E.Knuth, *The art of computer programming, Volume 1, Fundamental Algorithms*, Addison-Wesley, 1968.
- [13] D. Levin, Y. Peres and Wilmer, E., *Markov Chains and Mixing Times*, AMS, Providence RI, 2009.
- [14] L. Lovász. Random walks on graphs: a survey. *Bolyai Society Mathematical Studies*. Combinatorics, Paul Erdős is Eighty 2:1-46, Keszthely, Hungary, 1993.
- [15] T. Wassmer, Phase transition for the vacant set left by random walk on the giant component of a random graph. To appear in *Ann. Inst. H. Poincaré Probab. Statist.* [arXiv:1308.2548](#)
- [16] D. Windisch, Logarithmic components of the vacant set for random walk on a discrete torus, *Electronic Journal of Probability*, 13 (2008) 880–897.